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# Effective action of dressed mean fields for $\mathcal{N} = 4$ super-Yang–Mills theory

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**Abstract**

Based on general considerations such as $R$-operation and Slavnov-Taylor identity we show that the effective action, being understood as Legendre transform of the logarithm of the path integral, possesses particular structure in $\mathcal{N} = 4$ supersymmetric Yang–Mills theory for kernels of the effective action expressed in terms of the dressed effective fields. These dressed effective fields have been introduced in our previous papers as actual variables of the effective action. The concept of dressed effective fields naturally appears in the framework of solution to Slavnov–Taylor identity. The particularity of the structure is the independence of these kernels on the ultraviolet regularization scale $\Lambda$. These kernels are functions of mutual spacetime distances and of the gauge coupling. The fact that $\beta$ function in this theory is zero is used significantly.

Keywords: $R$-operation, gauge symmetry, $\mathcal{N} = 4$ supersymmetry, Slavnov–Taylor identity.

Slavnov–Taylor (ST) identity [1] is an important tool in quantum field theory. It is a consequence of BRST symmetry [2] of the tree level action of gauge theories, and it consists in an equation written for a functional that is called effective action [3]. An approach to solving the ST identity in gauge theories has been proposed recently [4, 5]. In this Letter we re-consider our analysis for a particular case of $\mathcal{N}=4$ supersymmetric theory. Our analysis will be based on five theoretical tools: $R$-operation [6], gauge symmetry, $\mathcal{N}=4$ supersymmetry, the ST identity itself, and absorbing two point Green's functions into a re-definition of the effective fields. Effective fields are variables of the effective action [3].

$\mathcal{N}=4$ super-Yang–Mills theory is widely considered from the point of view AdS/CFT correspondence [7]. Anomalous dimensions of gauge invariant operators are related to energies of string states [8]. In this Letter we consider $\mathcal{N}=4$ super-Yang–Mills theory from a different point. We analyse one particle irreducible (proper) correlators of this theory, which are kernels of the effective action. For example, a kernel can be proper vertex of several gluons. We hope this analysis can have application to calculation of maximal helicity violating amplitudes of processes with $n$ gluons [9, 10]. $\mathcal{N}=4$ super-Yang–Mills theory is useful theoretical playground to understand better the problems that stand in QCD. This model has special particle contents. In addition to one gluon and four Majorana fermions it contains six scalar fields. All the particles are in the adjoint representation of $SU(N)$ gauge group.

It has been known for a long time that for $\mathcal{N}=4$ Yang-Mills supersymmetric theory beta function of gauge coupling vanishes [11, 12, 13, 14]. We extensively use this fact in our analysis. Basic notation of this Letter coincides with notation of Ref. [4]. The main ST identity is [3]

$$\mathrm{Tr}\left[\int d\,x\,\frac{\delta\Gamma}{\delta A_m(x)}\frac{\delta\Gamma}{\delta K_m(x)}+\int d\,x\,\frac{\delta\Gamma}{\delta c(x)}\frac{\delta\Gamma}{\delta L(x)}-\int d\,x\,\frac{\delta\Gamma}{\delta b(x)}\left(\frac{1}{\alpha}\,\partial_m\,A_m(x)\right)\right] + \int d\,x\,\frac{\delta\Gamma}{\delta\phi(x)}\,\frac{\delta\Gamma}{\delta k(x)}+\int d\,x\,\frac{\delta\Gamma}{\delta\bar{k}(x)}\,\frac{\delta\Gamma}{\delta\bar{\phi}(x)}=0. \qquad (1)$$

The effective action is a functional of all the effective fields and external sources participating in this equation, $\Gamma\equiv\Gamma[A_m,b,c,\phi,\bar{\phi},K_m,L,k,\bar{k}]$ [22, 3]. The external sources $K_m,L,k,\bar{k}$ are coupled in the exponential of the path integral to the BRST transformations of fields from the measure of the path integral [3], that is, to the BRST transformations of fields $A_m,c,\phi,\bar{\phi}$ of the tree level action, respectively. The effective fields $A_m,b,c,\phi,\bar{\phi}$ are traditionally named by the same letters as are the fields $A_m,b,c,\phi,\bar{\phi}$ which are variables of the path integral. The effective fields are defined as variational derivatives of the logarithm of the path integral with respect to the corresponding external sources coupled to these variables of integration in the path integral [3]. The matter effective field $\phi$ stands for spinors as well as for scalars. We assume summations over all indices of the representation of matter fields. The traditional Lorentz gauge fixing is taken and the corresponding Faddeev-Popov ghost action introduced according to the line of Ref. [13]. These terms break supersymmetry of the tree level action. The $\beta$ function is zero but anomalous dimensions of propagators are non-zero [13].

Consider the vertex $Lcc$. Here we do not specify arguments of the effective fields. It is the only vertex which is invariant with respect to the ST identity at the classical level. At the quantum level it transforms to the form

$$\langle Lcc\rangle\times\langle Lcc\rangle+\langle LccA\rangle\times\langle K_m\partial_m c\rangle=0. \qquad (2)$$

This is a direct consequence of the main ST identity (1) and is a schematic form of the ST identity relating the $Lcc$ and $LccA$ field monomials. The precise form of this relation can be obtained by differentiating the identity (1) with respect to $L$ and three times with respect to $c$ and then by setting all the variables of the effective action equal to zero. The brackets in (2) mean that we have taken functional derivatives with respect to fields in the corresponding brackets at different arguments and then have put all the effective fields equal to zero.

We know from the theory of $R$-operation [6] that in Yang-Mills theory the divergences can be removed by re-defining the fields and the gauge coupling. Thus, there are four renormalization constants that multiply the ghost, gluon, spinor, and scalar fields [3]. The gauge coupling also must be renormalized but this is not the case in the theory under consideration. In this paper we concentrate on two regularizations: regularization by higher derivatives described in Ref. [3], $\Lambda$ is the regularization scale, and regularization by dimensional reduction. The regularization by higher derivatives has been constructed for supersymmetric theories in Refs. [15, 16]. Having used this regularization, new scheme has been proposed in Refs. [17, 18, 19]. We assume here that the component analog of that scheme can be constructed. The regularization by higher derivatives provides strong suppression of ultraviolet divergences by introducing additional terms with higher degrees of covariant derivatives acting on Yang–Mills tensor into the classical action, which are suppressed by appropriate degrees of the regularization scale $\Lambda$. In addition to this, it is necessary to introduce a modification of the Pauli–Villars regularization to guarantee the convergence of the one-loop diagrams [3]. This scheme does not break gauge invariance beyond one loop level. Moreover, it has been suggested in Ref. [3] that such a modification by Pauli–Villars terms to remove one-loop infinities is gauge invariant by construction. To regularize the fermion cycles, the usual Pauli–Villars regularization can be used. However, when applied to explicit examples, this approach is known to yield incorrect results [20]. A number of proposals have been put forward to emend this problem [21]. However, all these proposals contain as intermediate steps some non-trivial extensions of the original setup. For instance they either require intermediate dimensional regularization or include non-local terms in the action. Because of these problems we analyse the theory in the regularization by the dimensional reduction in a parallel way.

At one-loop level the part associated with the divergence of $Lcc$ term must be invariant itself under the ST identity since the second term in identity (2) is finite in the limit of removing regularization $\Lambda \to \infty$ [4]. According to Ref. [4], this results in the following integral equation for the part of the correlator $Lcc$ corresponding to the superficial divergence $\sim \ln \frac{p^2}{\Lambda^2}$ :

$$\int dx\ \Gamma_\Lambda(y',x,z')\Gamma_\Lambda(x,y,z) = \int dx\ \Gamma_\Lambda(y',y,x)\Gamma_\Lambda(x,z,z')$$
$$= \int dx\ \Gamma_\Lambda(y',x,z)\Gamma_\Lambda(x,z',y), \qquad (3)$$

where $\Gamma_\Lambda(x,y,z)$ is this scale($\Lambda-$) dependent part of the most general parametrization $\Gamma(x,y,z)$ of the correlator $Lcc$,

$$\Gamma \sim \int dx\ dy\ dz\ \Gamma(x,y,z) f^{abc} L^a(x) c^b(y) c^c(z). \qquad (4)$$

Here $f^{abc}$ is the group structure constant. The only solution to the integral equation (3)

is [4]

$$\Gamma_\Lambda(x,y,z) = \int dx'\ G_c(x'-x)\ G_c^{-1}(x'-y)\ G_c^{-1}(x'-z). \tag{5}$$

The subscript $\Lambda$ means scale-dependent part of the correlator. As can be seen, all scale-dependence of this correlator is concentrated in the dressing function. The complete correlator $Lcc$ at one loop level can be then written as

$$\int dx\ dy\ dz\ \Gamma(x,y,z)\frac{i}{2}f^{bca}L^a(x)c^b(y)c^c(z) =$$
$$= \int dx'dy'dz'dxdydz\ \tilde{\Gamma}(x',y',z')G_c(x'-x)\ G_c^{-1}(y'-y)\ \times$$
$$\times\ G_c^{-1}(z'-z)\frac{i}{2}f^{bca}L^a(x)c^b(y)c^c(z).$$

Here $\tilde{\Gamma}(x',y',z')$ is scale-independent kernel of $Lcc$ correlator [1].

We absorb this dressing function $G_c$ into the corresponding re-definition of the fields $L$ and $c$, and then divide the ghost propagator in two parts one of which is related to the dressing function of the ghost field $G_c$ and another we call the dressing function of the gluon field $G_A$. The effective field $K$ and the antighost field $b$ get opposite re-definition by integrating with the dressing function $G_A^{-1}$ [4]. The important point here is covariance of the part of the ST identity without gauge fixing term with respect to such redefinitions [4]. In terms of the dressed effective fields we have a useful relation which is consequence of the main ST identity (1) and can be obtained by differentiating the main ST identity two times with respect to $\tilde{c}$ and one time with respect to $\tilde{b}$. This resulting identity is

$$\left\langle \tilde{A}_m\tilde{b}\tilde{c}\right\rangle \times \left\langle \tilde{K}_m\tilde{c}\right\rangle + \left\langle \tilde{L}\tilde{c}\tilde{c}\right\rangle \times \left\langle \tilde{b}\tilde{c}\right\rangle = 0. \tag{6}$$

Again, this identity is written in a schematic way. However, a new important point appears here. Namely, since the two-point proper functions in terms of dressed effective fields are trivial tree level two-point proper functions, the divergences of $\left\langle \tilde{A}_m\tilde{b}\tilde{c}\right\rangle$ and $\left\langle \tilde{L}\tilde{c}\tilde{c}\right\rangle$ coincide. Since $\left\langle \tilde{L}\tilde{c}\tilde{c}\right\rangle$ is scale-(or $\Lambda-$) independent, the $\left\langle \tilde{A}_m\tilde{b}\tilde{c}\right\rangle$ is scale-independent also. Concerning the gluon propagator, one part of the divergence is in the dressing function $G_A$, and the rest of divergence would be absorbed in the re-definition of the gauge coupling constant. This last divergence is absent in $\mathcal{N}=4$ theories. All the other correlators are solved by the ST identity in terms of the dressed effective fields and their kernels are finite (do not posses divergence in the limit of removing regularization) and scale-independent.

Infinite parts of the dressing functions will be one loop counterterms corresponding to the re-definition of the fields. Then we can repeat this procedure at two loop level and so on, up to any order in loop number. Indeed, the re-definition by multiplication of the fields of the tree level action results in re-defining external legs of proper correlators in comparison with the unrenormalized theory. This property has been used by Bogoliubov and Shirkov [6] in the derivation of renormalization group equations. Re-defining the effective fields by dressing functions does not bring new aspects in this sense. Indeed, re-defining variables of the path integral by dressing will result in the dressing of the external

[1] In Ref.[4] we conjectured, based on Eqs.(3) and (4), that the complete $\Gamma(x,y,z)$ has the structure (5); this would correspond to $\tilde{\Gamma}(x',y',z') \propto \delta(x'-y')\delta(x'-z')$.

legs of the proper correlators. By proper correlators we mean kernels of $\Gamma$, that is, these kernels are one particle irreducible diagrams. We have already used this re-definition in the dressing functions in the $\tilde{L}\tilde{c}\tilde{c}$ correlator. Thus, new superficial divergences will have to satisfy the integral equation (3) at two loop order too in terms of effective fields $\tilde{L}\tilde{c}\tilde{c}$ which are effective fields dressed by one loop dressing function $G_c$. We then repeat this re-definition in the same procedure in each order of the perturbation theory as we did in the previous paragraph at one loop level.

Until now the pure gauge sector has been considered. Fermions are necessary for providing supersymmetry. Consider vertex $kc\phi$ at one loop level. The superficial divergence of this vertex is canceled by the divergence of the vertex $Lcc$. This means that in the part of the correlator $kc\phi$ corresponding to superficial divergence $\sim \ln \frac{p^2}{\Lambda^2}$ this divergence can be absorbed into the dressing functions in the following way:

$$\int dx dx_1 dy_1 dy_2 \; G_\phi(x-x_1)\; G_c^{-1}(x-y_1)\; G_\phi^{-1}(x-y_2)\; k(x_1)\; c(y_1)\; \phi(y_2). \tag{7}$$

Note that the dressing functions for the fermions and scalars $G_\phi^{-1}(x)$ are not fixed yet. We set them equal to halves of the two point matter functions. The rest of the vertices is restored in the unique way because the ST identity works.

This theory has intrinsic on-shell infrared divergences, like those canceled by bremsstrahlung of soft gluons (such a cancellation happens on shell). To regularize these divergences we can introduce mass parameter $\mu$ [23]. Such a trick breaks the ST identity by terms dependent on $\mu$. At the end the dependence on $\mu$ will disappear in physical matrix elements. We mean by the physical matrix element connected diagram on-shell contribution to amplitudes of particles. However, one can think that in the effective action (off shell) they could be present since cancellation of the infrared divergences happens (on shell) between proper and one particle reducible graphs [23]. Below we will indicate that off shell these infrared divergences do not exist at all in the position space[2].

In principle, infrared divergences in the effective action represent an outstanding problem that is not treated in the present work in necessary details. We show in this paragraph that in the regularization by the dimensional reduction this problem does not appear. It is enough to show this for the $Lcc$ correlator, because other correlators can be expressed in terms of that one by ST identity if we work in terms of dressed effective fields. The one-loop contribution depicted in Fig. 1 (the only diagram that can be drawn) is apparently convergent in the Landau gauge. The point is that the derivatives can be integrated out of the graph due to the property of transversality of the gluon propagator in the Landau gauge what makes immediately this graph convergent in the ultraviolet region but is safe in the infrared. Note that in other gauges this correlator remains divergent in the ultraviolet, and its scale dependence is contained in the dressing function $G_c$. The two-loop diagrams (planar) are drawn below in Fig. 2. The first two diagrams are apparently convergent in ultraviolet in Landau gauge since all subgraphs are convergent. This is due to the property of transversality that allows to integrate out the derivatives again. Infrared region is also not dangerous since even in Landau gauge the gluon propagator is safe in the infrared region.

The behavior of the theory in the IR region is not spoiled by the higher derivative regularization too. This is clear from the structure of the gluon propagator (in Landau

[2]We treat the theory in the position space. Infrared divergences are absent off shell also in the momentum space by the same reasons.

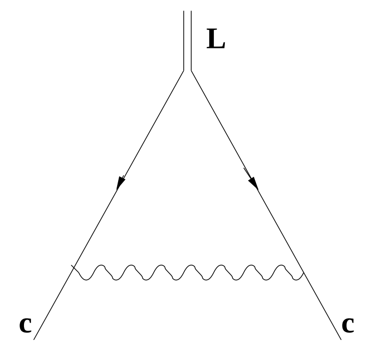


Figure 1: *One-loop contribution to the Lcc vertex. The wavy lines represent the gluons, the straight line are for the ghosts.*

gauge, for example) [3]:

$$D^{ab}_{\mu\nu} = \delta^{ab} \left[ - \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{1}{k^2 + k^6/\Lambda^4} \right].$$

Sixth degree of momentum in the denominator improves significantly ultraviolet behavior but in the infrared it is negligible in comparison with second degree of momentum. The infrared divergence appears when we en force put the on-shell condition $p_i^2 = 0$, where $p_i$ are external momenta. In general, infrared region is not dangerous off shell in component formulation in Wess-Zumino gauge when we regularize the theory by higher derivatives.

In such way we come to our main conclusion in this Letter. Namely, $\mathcal{N} = 4$ supersymmetric theory has scale-independent effective action in terms of the dressed effective fields (once we disregard the possible dependence on the infrared regulator mass which is usually taken to be infinitesimally small). All the dependence on the dimensionful parameter of ultraviolet regularization remains in the dressing functions only. This is in correspondence with direct calculation of anomalous dimensions and beta function in $\mathcal{N} = 4$ theory [13]. At one-loop level, kernels of this scale-independent theory are in general dilogarithms in momentum space. These dilogarithms are Fourier transforms of the kernels in position space as given below. For example, the correlator of the dressed effective fields $\tilde{L}$, $\tilde{c}$, and $\tilde{c}$ at one loop level in any $SU(N)$ gauge theory has, among others, the following contribution:

$$\left\langle \tilde{L}^a(x) \tilde{c}^b(y) \tilde{c}^c(z) \right\rangle \sim g^2 N \frac{1}{((z-y)^2)^2 (x-y)^2 (z-x)^2} f^{abc}, \tag{8}$$

where the dressed effective fields are made of undressed effective fields convoluted to the dressing functions. The latter are unspecified but they are parts of the two point proper Green functions. The terms of the type (8) can be obtained, for example, by calculating the one-loop $Lcc$ Green function in the Landau gauge (where $G_c(x) = \delta^{(4)}(x)$) and then using repeatedly the identity

$$\frac{1}{(2\pi)^4} \int d^4k \frac{e^{-ikx}}{k^2 + i\epsilon} = \lim_{\eta \to +0} \frac{i}{4\pi^2} \frac{1}{[(|x^0| - i\eta)^2 - \mathbf{x}^2]}.$$

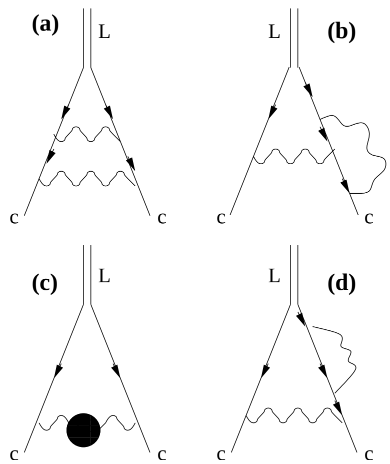


Figure 2: *Two-loop diagrams for the Lcc vertex. The wavy lines represent the gluons, the straight lines the ghosts. The black disc in (c) is for one-loop contribution in the renormalization of the vector propagator from scalar, spinor and ghost fields*

By the ST identity the correlators (8) are related to the vertex $KAc$ which in its turn is related to the tree gluon vertex $AAA$. The relation of these vertices is dictated by the ST identity and can be explicitly verified. Thus, the contribution similar to (8) can be found in the proper correlator of the three dressed gluons at one-loop level $< \tilde{A}^a_\mu(x)\tilde{A}^b_\nu(y)\tilde{A}^c_\lambda(z) >$ . At the same time, in $\mathcal{N} = 4$ supersymmetry we do not need to make additional renormalization in two point gluon function to absorb the rest of infinities from it into renormalization of the gauge coupling, since the $\beta$ function is zero.

We found that the dressed effective fields are the actual variables of the effective action. The effective action is to be written in terms of these dressed effective fields. In general, in non-supersymmetric gauge theory like QCD the dependence on UV regularization scale will be present inside the correlators of the dressed effective fields because it is necessary to remove the dependence on this scale by renormalization of the gauge coupling constant. In $\mathcal{N} = 4$ supersymmetric theory such a renormalization does not take place. Thus, the kernels for the dressed fields do not depend on scale. This might make possible an analysis of these kernels by tools of conformal field theory in all orders of perturbation theory.

We have shown in this Letter that such a scale independent structure of correlators is a direct consequence of the Slavnov–Taylor identity and it is encoded in the $Lcc$ correlator of the dressed effective fields. In general, by solving step-by-step ST identity it is possible to reproduce structure of all $n$-gluonic proper correlators in terms of dressed effective fields. Here the question how to define the concept of scattering can arise. The knowledge of correlators is not enough to define a scattering matrix. Indeed, it follows by the above construction that these correlators of dressed mean fields in $\mathcal{N} = 4$ supersymmetry do not have dependence on any mass parameter, or, stated otherwise, the theory in terms of dressed mean fields is conformal invariant. It is known that $S$-matrix in conformal field theory cannot be constructed, since we do not have any dimensional parameter like mass or scale to define scattering concepts like typical scattering length, or size of a meson and so on. The only observables in this theory are correlators of the gauge invariant operators

and their anomalous dimensions. However, on shell when we go to the amplitudes the scale appears due to infrared on shell divergences, so that scattering concepts can be introduced in a traditional way [24].

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